Fuel-Optimal Rendezvous Near a Point in General Keplerian Orbit

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Based on the linearized equations of motion of a spacecraft near a satellite in general Keplerian orbit and the assumptions of a bounded thrust magnitude and constant exhaust velocity, a fixed-duration, fuel-optimal rendezvous problem is formulated and is investigated for the constant-mass case through the solution for the primer vector, which is shown to satisfy the original differential equations that describe the spacecraft motion during unpowered flight. It is shown that there are no singular solutions to this rendezvous problem for noncircular Keplerian orbits and that consequently all optimal solutions for orbit eccentricities greater than zero are constructed from a finite number of intervals of full thrust and coast where the switches are determined from the primer vector.

I. Introduction

A NUMBER of early researchers contributed to the beginnings of the field of Optimal Space Trajectories. Goddard's problem of determining the method for a rocket to reach high altitude with minimum fuel expenditure, which was posed in 1919,[1] clearly falls into this field, and was later analyzed by Leitman[2,3] using the calculus of variations. Hohmann's problem[4] of the fuel-optimal transfer of a rocket between two coplanar circular orbits was presented in 1925 with an assumed solution composed of instantaneous initial and terminal thrusts defining an elliptical coasting orbit tangent to both circles. A number of papers on this and similar problems were generated (see the surveys, Refs. 5–8), but a fixed number of instantaneous impulses were usually assumed, so that the problem was one of parameter optimization, and the known methods of the calculus of variations were not applied. In the forties Contensou[9,10] developed optimization methods to analyze a class of planar problems of determining the optimal motion of aerospace vehicles through a resisting atmosphere. In the fifties Lawden produced a number of papers on the general problem of the fuel-optimal trajectories of a rocket in a Newtonian gravitational field (see Refs. 5–8), and many of his methods and results are unified in his book,[11] which relies on the calculus of variations as the tool for optimization. Leitmann[12,13] and Edelbaum[14–16] contributed significantly to similar problems.

These early contributions to the field are representative but far from exhaustive. A vast number of other works can be found from the surveys.[5–8] Later approaches to this field can be found in the book by Marec,[17] which uses the methods of optimal control theory, some of which are found in the previously mentioned early works of Contensou.

The development of the mathematical theory of optimal control, especially the principle of Pontryagin,[18] made available powerful tools for the investigation of optimal space trajectories. Within this framework, new approaches to questions of the existence of optimal solutions and sufficiency conditions and the investigation and significance of singular solutions were made possible. For example, Pontryagin's principle establishes both the necessary and sufficient conditions for fuel-optimal trajectories in linearized problems because the integrand of the cost function is convex in the state (Ref. 19, Chapter 5).

The present work uses optimal control theory to investigate the fixed-time, bounded thrust, fuel-optimal rendezvous of a spacecraft using direct linearization about a satellite in general Keplerian orbit. Many other approaches to this type of problem and the related transfer problem use a linearization with respect to the orbital parameters associated with the spacecraft. The idea of variable orbital elements and the introduction of characteristic velocity instead of time as the independent variable was apparently used for time-open cases first by Contensou,[20] followed by Breakwell,[21] and then very successfully by Marchal[22] and many others (see Refs. 5, 7, and 17). The direct approaches have also been useful, especially for linearized rendezvous problems in which the satellite orbit is circular or near-circular,[23–28] using equations of motion developed independently by Wheelon,[29] Clohessy and Wiltshire,[30] and Geyling.[31]

Extensions of these linearized equations to include elliptical satellite orbits were done by DeVries[32] (although the third equation contains an error), Tschauner and Hempel,[33] Shulman and Scott,[34] and for the planar case, Euler.[35] As we shall show, the equations found by these researchers are also valid for general Keplerian orbits and provide the means for a direct approach to the linearized rendezvous problem near a satellite in Keplerian orbit of arbitrary eccentricity. Each of the mentioned researchers presented analytical solutions for spacecraft trajectories associated with their specific investigations based on these equations, as did Euler and Shulman[36] and Tschauner.[37] A much earlier solution of the homogeneous form of these equations for arbitrary eccentricity, however, was found by Lawden,[38] but Lawden's solution described the primer vector during the coast phase of a fuel-optimal trajectory of a spacecraft in a Newtonian gravitational field. The differential equations describing Lawden's primer during coast also define the linearized spacecraft equations relative to a satellite in Keplerian orbit during unpowered flight and describe the primer vector associated with this linearized fuel-optimal rendezvous problem for the entire flight as well.

We see that Lawden's equations have a variety of interpretations and applications. Viewed as the solution of the linearized

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equations of relative motion, they can be used to generalize previous work to describe the relative motion of particles in separate elliptical orbits.[39,40] They were used by Eckel[41] just as defined by Lawden to solve the problem of fuel-optimal orbital transfer, assuming instantaneous impulses. We use them here as solutions to the primer of the linearized fuel-optimal rendezvous problem with bounded thrust to extend previous results from circular satellite orbits[25] to general Keplerian orbits and complement the work of Tschauner,[37] who investigated this problem by assuming two separate engines with a cost function for each engine effectively reducing the problem to a planar rendezvous problem.

We also investigate the possibility of singular solutions using the same general approach as in previous work,[25] and show rigorously that there are none for satellite orbits having an eccentricity greater than zero, a result found also in Marec's book (Ref. 17, Chapter 7) for linearized transfer problems using orbital elements. A similar conclusion was made by Hempel[42] using a simpler and less exact set of equations. For satellites having a circular orbit, some of the singular solutions were known in 1969 by Prussing[23] for the related problem in which the thrust of the spacecraft is unbounded; all cases of singular solutions are in Marec's book, and a recent study[43] classifies and discusses the nature of the singular solutions in detail. Since there are no singular solutions for noncircular Keplerian orbits, we can conclude that for these cases, all fuel-optimal trajectories consist of full thrust and coast, and these intervals are precisely defined by the primer vector, which is found in terms of elementary functions.

II. The Linearized Equations

We derive the equations of motion of a spacecraft relative to a satellite in Keplerian orbit about a planet as follows. At time t, the position of the satellite with respect to the planet is denoted by the vector $R(t)$, and the position of the spacecraft measured from the satellite is $r(t)$, so that the position of the spacecraft relative to the planet is $R(t) + r(t)$. The spacecraft is assumed to have a scalar point mass $m(t)$ and an applied thrust vector $T(t)$ and is acted on by a Newtonian gravitational force directed toward the center of the planet. In this paper, all vectors are elements of three-dimensional Euclidean space. For brevity we will not write the argument of a function in this section unless the context requires that we emphasize it. Using the symbol $|\,|$ to indicate the magnitude or Euclidean norm of a vector and a dot for differentiation with respect to t, the equation of motion of the spacecraft is

$$\ddot{R} + \ddot{r} = -\mu(R + r)/|R + r|^3 + T/m \tag{1}$$

where μ is the universal gravitational constant times the mass of the planet. Since the satellite is assumed to have no external force acting on it, its motion is Keplerian:

$$\ddot{R} = -\mu R/|R|^3 \tag{2}$$

The motion of the system is totally governed by Eqs. (1) and (2). Using Eq. (2), we can eliminate $\ddot{R}$ in Eq. (1), obtaining an equation of motion of the spacecraft relative to the satellite. Replacing $|R + r|^3$ by $|R|^3(1 + 2R \cdot r/|R|^2 + |r|^2/|R|^2)^{3/2}$ in Eq. (1), where the dot is used to indicate the scalar product, and linearizing the resulting expression, we obtain the following linear approximation describing the motion of the spacecraft relative to the satellite:

$$\ddot{r} = -\frac{\mu}{|R|^3}\left(r - 3\,\frac{R \cdot r\, R}{|R|^2}\right) + \frac{T}{m} \tag{3}$$

where the function R is obtained from Eq. (2) and the initial conditions. The solution of Eq. (2) is confined to a plane and defines the well-known Keplerian orbit

$$|R| = \frac{L^2}{\mu}(1 + e\cos\theta)^{-1} \tag{4}$$

where the angle $\theta(t)$, the true anomaly, is measured from the perigee, the eccentricity e is greater than or equal to zero, and L is the magnitude of the constant angular momentum divided by the mass of the satellite. We see from Eq. (4) that we must have $L \neq 0$ to have a nondegenerate orbit. The fact that the angular momentum is constant is expressed by

$$|R|^2\,\dot{\theta} = L \tag{5}$$

We now transform Eq. (3) from an inertial frame fixed in the planet to a rotating frame fixed in the satellite. We denote the vector angular velocity of the satellite by $\Omega(t)$. The transformed equation of motion of the spacecraft is

$$\ddot{r} + 2\Omega \times \dot{r} + \Omega \times (\Omega \times r) + \dot{\Omega} \times r = -\frac{\mu}{|R|^3}\left(r - 3\,\frac{R \cdot r\, R}{|R|^2}\right) + \frac{T}{m} \tag{6}$$

where the symbol $\times$ is used to indicate the vector product and the measurements of r, $\dot{r}$, and $\ddot{r}$ are all with respect to the rotating coordinate system fixed in the satellite. From this point on, we shall use the notation x instead of r when referring to the position vector of the spacecraft relative to this coordinate system. For this coordinate system, the positive direction of the X_1 axis is opposed to the motion of the satellite and perpendicular to the X_2 axis, whose positive direction is that of R, and the X_3 axis completes a right-handed system. We let $\omega(t) = |\Omega(t)| = \dot{\theta}(t)$ and $k = \mu/L^{3/2}$, eliminate $|R|^3$ in Eq. (6) by using Eq. (5), and write the components of Eq. (6) in the rotating coordinate system using the appropriate subscripts to obtain the following system of equations:

$$\ddot{x}_1 = (\omega^2 - k\omega^{3/2})x_1 + \dot{\omega}x_2 + 2\omega\dot{x}_2 + T_1/m$$

$$\ddot{x}_2 = -\dot{\omega}x_1 + (\omega^2 + 2k\omega^{3/2})x_2 - 2\omega\dot{x}_1 + T_2/m$$

$$\ddot{x}_3 = -k\omega^{3/2}x_3 + T_3/m \tag{7}$$

It follows from Eq. (5) that the true anomaly θ is a strictly increasing function of t. Because of this one-to-one correspondence, we find it convenient to change the variable from t to θ. Using a prime to indicate differentiation with respect to θ and noting that for a general function ϕ of t we have $\dot{\phi} = \omega\phi'$ and $\ddot{\phi} = \omega^2\phi'' + \omega\omega'\phi'$, the system of Eqs. (7) becomes

$$\omega^2x_1'' + \omega\omega'x_1' = (\omega^2 - k\omega^{3/2})x_1 + \omega\omega'x_2 + 2\omega^2x_2' + T_1/m$$

$$\omega^2x_2'' + \omega\omega'x_2' = -\omega\omega'x_1 + (\omega^2 + 2k\omega^{3/2})x_2 - 2\omega^2x_1' + T_2/m$$

$$\omega^2x_3'' + \omega\omega'x_3' = -k\omega^{3/2}x_3 + T_3/m \tag{8}$$

We note from Eqs. (4) and (5) that

$$\dot{\theta} = \omega = \frac{\mu^2}{L^3}(1 + e\cos\theta)^2 \tag{9}$$

This expression can never be zero, as can be seen from Eq. (5) and the fact that L is nonzero. Dividing each equation of Eqs. (8) by ω, substituting Eq. (9) and its derivative ω' into the resulting equations, and noting the definition of k, we obtain

$$(1+e\cos\theta)x_1''-2e\sin\theta x_1'=e\cos\theta x_1-2e\sin\theta x_2$$
$$+2(1+e\cos\theta)x_2'+a_1$$
$$(1+e\cos\theta)x_2''-2e\sin\theta x_2'=(3+e\cos\theta)x_2+2e\sin\theta x_1$$
$$-2(1+e\cos\theta)x_1'+a_2$$
$$(1+e\cos\theta)x_3''-2e\sin\theta x_3'=-x_3+a_3 \qquad (10)$$

where the adjusted acceleration vector $a(\theta)$, whose components are found in the preceding expression, is given by

$$a=\frac{1+e\cos\theta}{\omega^2}\frac{T}{m}=\frac{L^6}{\mu^4}\frac{T/m}{(1+e\cos\theta)^3} \qquad (11)$$

Except for the differences in the directions of the coordinate axes and in the applied acceleration terms, Eqs. (10) is the same as the equations found by Shulman and Scott (Ref. 34, Eq. 2).

Given the function T/m, one can solve the linear nonhomogeneous system of Eqs. (10) and (11) to obtain the coordinates x_1, x_2, and x_3 in terms of the orbital angle θ. Generally, the mass m depends on the thrust T. If we assume a constant exhaust velocity, then $\dot{m}(t)=-|T(t)|/c$ where c denotes the constant exhaust velocity. Changing the variable from t to θ, this equation becomes

$$m'=-\frac{|T|}{c\omega}=-\frac{L^3}{c\mu^2}\frac{|T|}{(1+e\cos\theta)^2} \qquad (12)$$

Given T as a function of the true anomaly θ, the mass of the spacecraft and its position relative to the coordinate system centered in the satellite are defined by the system of Eqs. (10–12). If T is given as a function of time t, as is the usual situation, we add Eq. (9), which defines the relationship between t and θ. For the Keplerian motion of the spacecraft in which the thrust is zero over an interval, the system reduces to Eqs. (10) with $a_1=a_2=a_3=0$.

For the Keplerian motion of the spacecraft, Eqs. (10) can be solved analytically. We note, however, from their form that a transformation is suggested; in fact they can be rewritten as

$$\frac{1}{1+e\cos\theta}[(1+e\cos\theta)^2x_1']'=e\cos\theta x_1$$
$$+2[(1+e\cos\theta)x_2]'+a_1$$
$$\frac{1}{1+e\cos\theta}[(1+e\cos\theta)^2x_2']'=(3+e\cos\theta)x_2$$
$$-2[(1+e\cos\theta)x_1]'+a_2$$
$$\frac{1}{1+e\cos\theta}[(1+e\cos\theta)^2x_3']'=-x_3+a_3 \qquad (13)$$

We find it convenient to introduce the vector $y=(y_1,y_2,y_3)$ by

$$y=(1+e\cos\theta)x \qquad (14)$$

With this transformation, the system of Eqs. (13) assumes the succinct form

$$y_1''=2y_2'+a_1$$
$$y_2''=\frac{3y_2}{1+e\cos\theta}-2y_1'+a_2$$
$$y_3''=-y_3+a_3 \qquad (15)$$

Other less detailed derivations of these equations can be found for elliptical orbits in Refs. 32–35. The homogeneous form of these equations first appeared in the work of Lawden[38] to describe the primer vector associated with the unpowered part of a fuel-optimal rocket trajectory in an inverse square law gravitational field.

III. Solutions of the Equations

If the original Eqs. (1) and (2) or the subsequent linearized version are defined over a fixed, closed-time interval $t_0\le t\le t_f$, then Eqs. (10–15) are also defined on a fixed, closed interval $\theta_0\le\theta\le\theta_f$, as can be seen from Eq. (9). In fact,

$$t_f-t_0=\frac{L^3}{\mu^2}\int_{\theta_0}^{\theta_f}\frac{d\theta}{(1+e\cos\theta)^2} \qquad (16)$$

where

$$-\cos^{-1}\left(-\frac{1}{e}\right)<\theta_0<\theta_f<\cos^{-1}\left(-\frac{1}{e}\right) \qquad e\ge 1 \qquad (17)$$

This integral can be evaluated to establish the relationship between the total flight time and the terminal and initial orbital angles of the satellite. Similarly, the instantaneous flight time is determined from the instantaneous and initial orbital angles as:

$$t-t_0=\frac{L^3}{\mu^2}[g(\theta)-g(\theta_0)]$$
$$g(\theta)=\frac{1}{1-e^2}\left[\frac{-e\sin\theta}{1+e\cos\theta}+\frac{2}{\sqrt{1-e^2}}\right.$$
$$\left.\times\tan^{-1}\left(\frac{\sqrt{1-e^2}}{1+e}\tan\frac{\theta}{2}\right)\right],\quad e<1$$
$$g(\theta)=1/2\tan(\theta/2)+1/6\tan^3(\theta/2),\quad e=1$$
$$g(\theta)=\frac{1}{1-e^2}\left[\frac{-e\sin\theta}{1+e\cos\theta}+\frac{2}{\sqrt{e^2-1}}\right.$$
$$\left.\times\tanh^{-1}\left(\frac{\sqrt{e^2-1}}{e+1}\tan\frac{\theta}{2}\right)\right],\quad e>1 \qquad (18)$$

The first equation of (18) defines a form of Kepler's equation. Changing the variable from θ to the eccentric anomaly will produce the usual form of Kepler's equation.

We now investigate the solution of Eqs. (15) on an interval $\theta_0\le\theta\le\theta_f$ satisfying Eq. (17). The third equation represents forced harmonic oscillations and is decoupled from the system. We can immediately get a first integral of the first equation and use this result in the second equation to eliminate $y_1'(\theta)$ so that the first two equations become

$$y_1'(\theta)=2y_2(\theta)+r_1(\theta)$$
$$y_2''(\theta)+\left(4-\frac{3}{1+e\cos\theta}\right)y_2(\theta)=r_2(\theta) \qquad (19)$$

where

$$r_1(\theta)=\int_{\theta_0}^{\theta}a_1(\zeta)\,d\zeta+c_1$$
$$r_2(\theta)=a_2(\theta)-2\int_{\theta_0}^{\theta}a_1(\zeta)\,d\zeta-2c_1 \qquad (20)$$

and c_1 is an arbitrary constant of integration. The complete solution of the system can be found if any nontrivial solution is found for the homogeneous linear differential equation

$$y_2''(\theta)+\left(\frac{1-4e\cos\theta}{1+e\cos\theta}\right)y_2(\theta)=0 \qquad (21)$$

Although this is a special case of Hill's equation, we can find a solution in terms of elementary functions. One finds that the function

$$\phi(\theta) = \sin\theta(1 + e\cos\theta) \tag{22}$$

satisfies this differential equation, and by the method of the reduction of order we find a second solution of Eq. (21) to be $\phi(\theta)I(\theta)$ where

$$I(\theta) = \int \frac{d\theta}{\sin^2\theta(1 + e\cos\theta)^2} \tag{23}$$

Although Somewhat expansive, this integral can be evaluated in terms of elementary functions. We list its evaluation up to an arbitrary constant of integration as

$$I(\theta) = -\cot\theta, \qquad e = 0$$

$$I(\theta) = \frac{6e^2}{(1-e^2)^{5/2}} \tan^{-1}\left(\frac{(e-1)G(\theta)}{(1-e^2)^{1/2}}\right) - E(\theta), \qquad 0 < e < 1$$

$$I(\theta) = \frac{-2\cos^3\theta + 4\cos^2\theta + \cos\theta - 2}{5\sin\theta(1+\cos\theta)^2}, \qquad e = 1$$

$$I(\theta) = \frac{3e^2}{(e^2-1)^{3/2}} \log\left|\frac{(e-1)G(\theta) - (e^2-1)^{1/2}}{(e-1)G(\theta) + (e^2-1)^{1/2}}\right| - E(\theta), \qquad e > 1 \tag{24}$$

where

$$G(\theta) = \frac{\sin\theta}{1 + \cos\theta}$$

$$E(\theta) = \frac{1}{2(e-1)^2} \times \left(\frac{(5e^3 - 3e^2 + 3e - 1)G(\theta)^2 - (e^2-1)(e-1)}{(e-1)(e+1)^2 G(\theta)^3 - (e+1)^3 G(\theta)} + G(\theta)\right) \tag{25}$$

This integral was evaluated in a different form by Lawden.[38] It is seen from each of the preceding four cases that the function $\phi(\theta)I(\theta)$ has removable singularities where $\sin\theta = 0$, so that this function may be regarded as analytic in the region of Eq. (17).

With this information, we find a particular solution of the nonhomogeneous second equation of Eqs. (19) to be

$$\phi(\theta) \int \frac{L(\theta)}{\phi(\theta)^2} d\theta$$

where

$$L(\theta) = \int r_2(\theta)\phi(\theta) d\theta \tag{26}$$

Having the complete solution of the second equation of Eqs. (19), we can find the complete solution of the first by integration. We therefore find the general solution of the system of Eqs. (15) for $e > 0$ to be

$$y_1(\theta) = -\frac{b_1}{e}(1 + e\cos\theta)^2 - \frac{b_2}{e}[(1 + e\cos\theta)^2 I(\theta) + \cot\theta] - \frac{1}{e}(1 + e\cos\theta)^2 \int \frac{L(\theta)}{\phi(\theta)^2} d\theta + \frac{1}{e} \int \frac{L(\theta)}{\sin^2\theta} d\theta + \int r_1(\theta) d\theta + c_2$$

$$y_2(\theta) = \sin\theta(1 + e\cos\theta)\left[b_1 + b_2 I(\theta) + \int \frac{L(\theta)}{\phi(\theta)^2} d\theta\right]$$

$$y_3(\theta) = \left[\alpha - \int a_3(\theta)\sin\theta d\theta\right]\cos\theta + \left[\beta + \int a_3(\theta)\cos\theta d\theta\right]\sin\theta \tag{27}$$

where b_1, b_2, c_2, α, and β are arbitrary constants. The case where $e = 0$ is considerably simpler referring to the case where the satellite is in circular orbit, and the solution in terms of $x(t)$, at least for the unpowered flight of the spacecraft, can be determined from information found in several of the references.[23,25,28–31]

For the case of unpowered flight $[a(\theta) = 0]$ over the interval $\theta_0 \le \theta \le \theta_f$, Eq. (27) reduces to

$$y_1(\theta) = -\frac{b_1}{e}(1 + e\cos\theta)^2 - \frac{b_2}{e}[(1 + e\cos\theta)^2 I(\theta) + \cot\theta] - \frac{c_1}{e}[\sin\theta + \phi(\theta)] + c_2$$

$$y_2(\theta) = \sin\theta(1 + e\cos\theta)[b_1 + b_2 I(\theta)] - \frac{c_1}{e}\cos\theta(1 + e\cos\theta)$$

$$y_3(\theta) = \alpha\cos\theta + \beta\sin\theta \tag{28}$$

These equations were found by Lawden[11,38] in a somewhat different form as a solution to the primer vector equations defined by the homogeneous form of Eqs. (15). They were found later[33,34] in the context of describing the position of a spacecraft near a satellite in elliptical orbit. Note that these solutions are analytic, since $\phi(\theta)I(\theta)$ is analytic everywhere in spite of the fact that $I(\theta)$ is not.

We can find the complete solution of Eqs. (10) describing the spacecraft position relative to a rotating coordinate frame fixed in the satellite using Eq. (14) by dividing the function y from Eqs. (27) or (28) by the expression $1 + e\cos\theta$. We observe that the solution of Eqs. (15) is not significantly simpler or easier to obtain than the solution of Eqs. (10). The primary advantage of the transformation Eq. (14) is the conciseness of the resulting Eqs. (15).

IV. Optimal Rendezvous of the Spacecraft

The fuel-optimal rendezvous problem is posed as follows. We are given real numbers θ_0 and θ_f with $\theta_0 < \theta_f$ and define the closed interval $\Theta = [\theta_0, \theta_f]$ that satisfies the restriction of Eq. (17). We are also given a positive real number T_m representing the maximum allowable magnitude of the thrust. The class of *admissible thrust functions* is defined as the set of all Lebesgue measurable functions T satisfying the bound $|T(\theta)| \le T_m$ a.e. on Θ. The positive number m_0 represents the initial mass of the spacecraft, the fixed vectors y_0 and v_0 represent the initial position and velocity of the spacecraft respectively after the transformation of Eq. (14), and similarly the vectors y_f and v_f represent the final position and velocity respectively. The fuel-optimal rendezvous problem is to find a thrust function T from the class of admissible thrust functions which maximizes the final mass of the spacecraft $m(\theta_f)$ subject to the differential Eqs. (12) and (15), which hold a.e. on Θ, where the function a is given by Eq. (11), the initial conditions

$$m(\theta_0) = m_0 \tag{29}$$

$$y(\theta_0) = y_0, \; y'(\theta_0) = v_0 \tag{30}$$

and the terminal conditions

$$y(\theta_f) = y_f, \; y'(\theta_f) = v_f \tag{31}$$

As is seen from Eq. (12), an equivalent formulation of the problem is to minimize the integral

$$J[T] = \int_{\theta_0}^{\theta_f} \frac{|T(\theta)| d\theta}{(1 + e\cos\theta)^2} \tag{32}$$

subject to the same restrictions.

We now confine ourselves to problems in which $L^3 T_m/(c\mu^2)$ and $\theta_f - \theta_0$ are small enough that the mass expended is very small compared with the initial mass m_0. We therefore disregard Eq. (12) and set $m(\theta) = m_0$, define the positive number $b = L^6 T_m / (\mu^4 m_0)$ and introduce the *control* function by $u(\theta) = T(\theta)/T_m$. With these changes we can replace $(L^6/\mu^4) T(\theta)/m(\theta)$ in Eq. (11) by $bu(\theta)$. The fact that the thrust is bounded is expressed by $|u(\theta)| \leq 1$. We put Eqs. (15) in state vector form, changing the notation in Eqs. (30) and (31), rewrite Eq. (32) in terms of u instead of T, and state the simplified fuel-optimal rendezvous problem as follows.

The set of Lebesgue measurable functions such that $|u(\theta)| \leq 1$ a.e. on Θ is called the set of *admissible controls*. We seek an admissible control that miminizes

$$J[u] = \int_{\theta_0}^{\theta_f} \frac{|u(\theta)| \mathrm{d}\theta}{(1 + e \cos\theta)^2} \tag{33}$$

subject to the differential equations

$$y'(\theta) = v(\theta)$$
$$v'(\theta) = A(\theta) y(\theta) + B v(\theta) + \frac{b}{(1 + e \cos\theta)^3} u(\theta) \tag{34}$$

which hold a.e. on Θ and the fixed-end conditions

$$y(\theta_0) = y_0, \ v(\theta_0) = v_0 \tag{35}$$

$$y(\theta_f) = y_f, \ v(\theta_f) = v_f \tag{36}$$

where the matrices $A(\theta)$ and B are as follows:

$$A(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{3}{1 + e \cos\theta} & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{37}$$

If there exists an admissible control such that Eqs. (34–37) are satisfied, then the well-known principle of Pontryagin provides both the necessary and sufficient conditions for the solution of the given optimal control problem because the differential Eqs. (34) are linear and the integrand of Eq. (33) is independent of the state (Ref. 19, Sec. 5.2). For this reason, the triple (y, v, u) is a minimizing solution of this optimal control problem if and only if it satisfies the conditions of Eqs. (34–37) and if the Hamiltonian defined by

$$H(y(\theta), v(\theta), l, p(\theta), q(\theta), u_0) = \frac{l |u_0|}{(1 + e \cos\theta)^2} + p(\theta)^T v(\theta)$$
$$+ q(\theta)^T \left[A(\theta) y(\theta) + B v(\theta) + \frac{b u_0}{(1 + e \cos\theta)^3} \right] \tag{38}$$

when viewed as a function of u_0 is minimized by $u(\theta)$ a.e. on Θ where $l \geq 0$ and the functions p and q are solutions of the adjoint differential equations

$$p'(\theta) = -A(\theta)^T q(\theta)$$
$$q'(\theta) = -p(\theta) - B^T q(\theta) \tag{39}$$

The superscript T is used to indicate that a matrix is transposed.

For end conditions in which $l = 0$, a minimizing solution is called *abnormal*; otherwise it is called *normal*. We shall restrict our attention to solutions that are normal, in which case l may be any positive number. We set $l = b$ for convenience.

We see from Eq. (38) that a minimizing control function $u(\theta)$ must minimize $\dfrac{b}{(1 + e \cos\theta)^2} \left(|u_0| + \dfrac{q(\theta)^T u_0}{1 + e \cos\theta} \right)$ a.e. on Θ. Consequently it is defined a.e. on Θ by

$$u(\theta) = \begin{cases} 0, & \dfrac{|q(\theta)|}{1 + e \cos\theta} < 1 \\ -\dfrac{q(\theta)}{|q(\theta)|}, & \dfrac{|q(\theta)|}{1 + e \cos\theta} > 1 \\ -\dfrac{q(\theta) f(\theta)}{1 + e \cos\theta}, & \dfrac{|q(\theta)|}{1 + e\cos\theta} = 1 \end{cases} \tag{40}$$

where f is any Lebesgue measurable function satisfying

$$0 \leq f(\theta) \leq 1 \tag{41}$$

If a minimizing control function u satisfies either or both of the first two conditions of Eq. (40) a.e. on a subset $S \subseteq \Theta$ of positive Lebesgue measure, then u is called *nonsingular* on S. If it satisfies the third condition of Eq. (40) a.e. on a subset $S \subseteq \Theta$ of positive Lebesgue measure where f is given by Eq. (41) a.e. on S, then it is called *singular* on S.

Equation (40) shows that an optimal control function that is nonsingular on a measurable set S either has unit magnitude (full thrust) or is zero (coast) a.e. on S. Whichever of these situations occurs or whether or not an optimal control function is singular depends on the function q, which we call the Lawden primer (Ref. 11, Chapter 3). By eliminating p in Eq. (39) and observing from Eq. (37) that $A(\theta) = A(\theta)^T$ and $B = -B^T$, we see that the primer satisfies the differential equation

$$q''(\theta) - B q'(\theta) - A(\theta) q(\theta) = 0 \tag{42}$$

We now compare this equation with the following differential equation obtained by eliminating $v(\theta)$ in Eqs. (34):

$$y''(\theta) - B y'(\theta) - A(\theta) y(\theta) = \frac{b}{(1 + e \cos\theta)^3} u(\theta) \tag{43}$$

It is clear that Eq. (42), which describes Lawden's primer, is the same differential equation as the homogeneous form of Eq. (43), which describes the transformed position of the spacecraft during unpowered flight. For this reason, the complete solution of Eq. (42) is immediately obtained from Eq. (28). As indicated earlier, this solution also describes Lawden's primer during the coasting (Keplerian) phase of an optimal rocket trajectory in a Newtonian gravitational field. In that context it could not be used to determine when to switch the engine off as it can here.

We find it convenient to transform Lawden's primer as follows:

$$Q(\theta) = q(\theta)/(1 + e \cos\theta) \tag{44}$$

With this transformation, the optimal thrusting logic of Eq. (40) becomes

$$u(\theta) = \begin{cases} 0, & |Q(\theta)| < 1 \\ -\dfrac{Q(\theta)}{|Q(\theta)|}, & |Q(\theta)| > 1 \\ -Q(\theta) f(\theta), & |Q(\theta)| = 1 \end{cases} \tag{45}$$

which holds a.e. on Θ where f satisfies Eq. (41). From the expression for q as determined from Eqs. (28) and from Eq.

(44), we see that Q is given by

$$Q_1(\theta) = -\frac{b_1}{e}(1 + e\cos\theta) - \frac{b_2}{e}\left[(1 + e\cos\theta)I(\theta) + \frac{\cos\theta}{\phi(\theta)}\right]$$
$$-\frac{c_1}{e}\frac{[\sin\theta + \phi(\theta)]}{1 + e\cos\theta} + \frac{c_2}{1 + e\cos\theta}$$
$$Q_2(\theta) = [b_1 + b_2 I(\theta)]\sin\theta - \frac{c_1}{e}\cos\theta$$
$$Q_3(\theta) = \frac{\alpha\cos\theta + \beta\sin\theta}{1 + e\cos\theta} \tag{46}$$

where $\phi(\theta)$ is given by Eq. (22) and $I(\theta)$ by Eq. (23) or Eqs. (24) and (25). As previously indicated, Q is analytic, despite its apparent form. It can be shown that Q is Lawden's primer associated with x, as described by the system of Eqs. (10), just as q is Lawden's primer associated with y, as described by the system of Eqs. (15).

The solution of the simplified fuel-optimal rendezvous problem is described by Eqs. (23–27) and (11), where $(L^6/\mu^4)T/m$ is replaced by $bu(\theta)$ as determined from Eqs. (45) and (46), where the constants of integration in Eqs. (27) and (46) are not necessarily the same although the same notation is used. The determination of these constants for the specified end conditions of Eqs. (35) and (36) must be accomplished in order to solve specific rendezvous problems.

V. Investigation for Singular Solutions

The solution of the fuel-optimal rendezvous problem is considerably simplified if it can be shown that there are no singular solutions, because the optimal thrusting function is then defined completely in terms of the first two conditions of Eq. (45) and we can assert that all optimal solutions are comprised exclusively of intervals of full thrust and coast. We shall show that this is indeed the case if $e > 0$. If $e = 0$, the problem degenerates to the simpler case in which the satellite is in circular orbit. This problem is known to have singular solutions, which have recently been discussed in detail.[43] Earlier studies on similar problems in which the linearization is based on orbital parameters show also that singular solutions occur about a satellite in Keplerian orbit only if $e = 0$; these results are summarized by Marec (Ref. 17, Chapter 7).

In order to establish a contradiction, let us assume that $e > 0$ and that an optimal solution is singular on a subset $S \subseteq \Theta$ of positive Lebesgue measure. The optimal control function u is therefore defined by the third condition of Eq. (45), where the transformed primer Q has unit magnitude a.e. on S so that

$$Q_1(\theta)^2 + Q_2(\theta)^2 + Q_3(\theta)^2 - 1 = 0 \tag{47}$$

a.e. on S. Upon substituting the expressions from Eq. (46) into this equation and squaring terms, the equation can be put in the form

$$a(\theta)I(\theta)^2 + b(\theta)I(\theta) + c(\theta) = 0 \tag{48}$$

where the coefficients $a(\theta)$, $b(\theta)$, and $c(\theta)$ are ratios of trigonometric polynomials. This notation is completely independent of earlier terminology.

We first consider the case where $e \neq 1$. It is seen from the evaluation of $I(\theta)$ through Eq. (24) that, in this case, $I(\theta)$ has an arctangent or logarithm term and is therefore not algebraic over the field generated by the trigonometric functions; consequently $a(\theta) = b(\theta) = c(\theta) = 0$ a.e. on S. The coefficient $a(\theta)$ is by far the most easily evaluated and is found to be given by

$$a(\theta) = b_2^2(1 + 1/e^2 + (2/e)\cos\theta) \tag{49}$$

The fact that $a(\theta) = 0$ a.e. on S implies that $b_2 = 0$, since the cosine function cannot be constant on a set of positive measure. Setting $b_2 = 0$ considerably simplifies the first two equations of Eqs. (46).

We next consider the case where $e = 1$. In this case, $I(\theta)$ is a ratio of trigonometric polynomials as seen from Eq. (24), and the preceding argument does not hold without modification. Substituting the expression for $I(\theta)$ into Eq. (46) and substituting this result into Eq. (47) leads to a ratio of trigonometric polynomials, the numerator of which must be zero a.e. on S. Using a computer program for symbol manipulation, this numerator was expanded in terms of the linearly independent functions $\cos^n\theta$, $n = 0, 1, \ldots, 8$, and $\sin\theta\cos^n\theta$, $n = 0, 1, \ldots, 7$. Setting the coefficient of each of these terms equal to zero establishes 17 algebraic relationships between the constants b_1, b_2, c_1, c_2, α, and β. Three of these relationships, namely the ones obtained from the coefficients of $\sin\theta\cos^7\theta$, $\cos^7\theta$, and $\cos^8\theta$ are sufficient to show that $b_2 = 0$.

Having established that $b_2 = 0$, we substitute the resulting Eqs. (46) into Eq. (47) and expand this in terms of the linearly independent functions $\cos^n\theta$, $n = 0, 1, 2, 3, 4$ and $\sin\theta\cos^n\theta$, $n = 0, 1, 2, 3$ using the computer program for algebraic manipulation. The resulting expression is zero a.e. on S, and by the linear independence of the terms, we can set the nine coefficients equal to zero. This establishes nine algebraic equations in the constants b_1, c_1, c_2, α, and β. The examination of these equations shows them to be inconsistent unless $e = 0$. This shows that singular solutions are impossible for $e > 0$.

Having found that there are no singular solutions for $e > 0$, we can assert that for noncircular Keplerian orbits of the satellite, the fuel-optimal normalized thrust of the spacecraft is defined by the logic

$$u(\theta) = \begin{cases} 0, & |Q(\theta)| < 1 \\ -Q(\theta)/|Q(\theta)|, & |Q(\theta)| > 1 \end{cases} \tag{50}$$

a.e. on Θ where $Q(\theta)$ is determined by Eqs. (46). Since Q is continuous, we see that Θ can be decomposed into the union of a relatively open set of zero applied thrust, a relatively open set of full thrust, and a closed set of Lebesgue measure zero where Eq. (47) is satisfied, which includes the switches between thrust and coast. Since Θ is bounded, it can be shown from the fact that the left-hand side of Eq. (47) is analytic that at most finitely many switches are possible, so that Θ can be decomposed into a finite number of relatively open intervals of full thrust and coast separated by isolated switches. The determination of the maximum number of switches given the closed interval Θ and arbitrary fixed-end conditions is an unsolved problem. Even for the case in which the satellite orbit is circular and $\theta_f - \theta_0 = 2\pi$, this problem is presently unsolved, although the maximum number of thrusting and coasting intervals is conjectured to be seven.[44]

VI. Conclusions

The structure of the solution of the fixed-duration linear fuel-optimal rendezvous problem of a spacecraft with a constant mass and bounded thrust near a satellite in arbitrary Keplerian orbit has been determined, generalizing previous work in which the satellite orbit is circular and complementing the work of Tschauner and Hempel for elliptical orbits. It is shown that Lawden's solution for the primer vector for coasting intervals found in 1954 also applies to this problem during coasting or thrusting intervals. A rigorous argument shows that no intermediate thrust arcs are possible for noncircular Keplerian orbits of the satellite and that any optimal solution can be constructed from a finite number of intervals of full thrust and coast.

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